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Graph Invariants of Vassiliev Type and Application to 4D Quantum Gravity

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Abstract

We consider a special class of Kauffman's graph invariants of rigid vertex isotopy (graph invariants of Vassiliev type). They are given by a functor from a category of colored and oriented graphs embedded into a 3-space to a category of representations of the quasi-triangular ribbon Hopf algebra $U_q(sl(2,C))$. Coefficients in expansions of them with respect to x ($q = e^x$) are known as the Vassiliev invariants of finite type. In the present paper, we construct two types of tangle operators of vertices. One of them corresponds to a Casimir operator insertion at a transverse double point of Wilson loops. This paper proposes a non-perturbative generalization of Kauffman's recent result based on a perturbative analysis of the Chern-Simons quantum field theory. As a result, a quantum group analog of Penrose's spin network is established taking into account of the orientation. We also deal with the 4-dimensional canonical quantum gravity of Ashtekar. It is verified that the graph invariants of Vassiliev type are compatible with constraints of the quantum gravity in the loop space representation of Rovelli and Smolin.

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1 Introduction

The concept of Vassiliev Invariants was originally introduced in the theory of the knot space from a side of the algebraic topology[36].Let M be a space of all smooth maps $S^1 \to S^3$ going through a fixed point and having a fixed tangent vector at the fixed point. M is connected. The knot space is defined by $M \backslash \Sigma$ where Σ is called the "discriminant", i.e., a set of all singular maps with multiple points or vanishing tangent vectors. Any equivalence class of knot embeddings by the ambient isotopy of S^3 corresponds to a connected component of $M \backslash \Sigma$. In this sense, all the knots are classified by the space $H_0(M \backslash \Sigma)$. Each connected component of the knot space is separated by walls which constitute the discriminant Σ.

The Vassiliev invariants are introduced as follows. Let $\Sigma_j \subset \Sigma$ be a space of j-embeddings, i.e., a space of all singular maps whose singularities are j transverse double points. Then there exists a natural filtration $\Sigma \supset \Sigma_1 \supset \Sigma_2 \supset \cdots \supset \Sigma_j \supset \cdots$. Using this stratification, Vassiliev considered a spectral sequence to compute $H^0(M \backslash \Sigma)$. He obtained an inclusion: $H^0(M \backslash \Sigma) \supset \cdots E_\infty^{-j,j} \supset E_\infty^{-(j-1),j-1} \cdots \supset E_\infty^{-1,1}$. Each stabilized limit $E_\infty^{-j,j}$ is called the Vassiliev invariant of order j. It means that any element of $E_\infty^{-j,j}$ vanishes whenever it is evaluated for singular maps with more than j transverse double points. In addition, Vassiliev conjectured $E_\infty^{-j,j} \otimes_Z K \cong E_1^{-j,j} \otimes_Z K$ for some rings K. After the pioneered work of Vassiliev, Birman and Lin[11] calculated $E_1^{-j,j}$ in a combinatorial way based on some axioms. In their discussions, they introduced a special class of functionals defined over a space of cord diagrams D^j. D^j represents a space of all configurations of j pairs of $2j$ points on S^1 connected by j cords. Inspired by the works of Vassiliev, Birman and Lin, Kontsevich[24] verified $E_\infty^{-j,j} \otimes_Z C \cong E_1^{-j,j} \otimes_Z C$ using an integral representation of the Vassiliev invariants. In the definition of Kontsevich's integral representation, the weight system[9]plays an important role. It is given by a space of maps from some kind of Hopf algebra[24] generated by the cord diagrams to a ring K.

The relation of the Vassiliev invariants to well-known quantum group invariants such as the HOMFLY polynomials and the Kauffman polynomials was investigated by Birman, Lin[11][25][26] and Puinikhin[29][30] to a great extent. It was pointed out by them that the Vassiliev invariants of finite type or the Vassiliev invariants of finite order play an important role. They appear as coefficients in expansions of the quantum group invariants with respect to $x$where $q = e^x$. How about the connection of the Vassiliev invariants to the Chern-Simons gauge field theory? In this respect, we can't forget to mention the work of E. Witten[37]. He shed a light on a relation between the quantum group invariants of links and the 2D conformal field theory and arrived at an observation that the quantum group invariants of links can be regarded as vacuum expectation values of linking Wilson loops in the CS (Chern-Simons) quantum gauge field theory. The Wilson loop is a trace of a holonomy along a link component. But his discussion was implicitly based on a premise that two different quantization methods of the CS gauge field theory, i.e., the canonical quantization and the path integral quantization, are equivalent. Some authors attempted to have a foundation to confirm themselves about the premise. A perturbative analysis of the path integral of the CS gauge field theory is one of ways to check it. Works of

Axelrod, Singer[4][5] and Bar-Natan[9] treating the CS perturbation in the context of the Vassiliev invariants are of interest to us. In particular, Bar-Natan uncovered a relation between the Vassiliev Invariants and the Feynman diagrams in the CS quantum gauge field theory. They are connected by the concept of the weight system.

In the present paper, we are interested in a recent work of Kauffman[19]. (The Vassiliev invariants given by the quantum group invariants of links can be regarded as a special class of graph invariants that he introduced a few years ago[18][19][20].) In [19], he discussed the perturbation theory of the CS quantum gauge field theory to obtain a path integral representation of the graph invariants of Vassiliev type in terms of the Wilson loops with transverse double points. It was observed by him that the Vassiliev Invariant given by a j-embedding can be represented by a CS vacuum expectation value of the Wilson loops with j transverse double points where the quadratic Casimir operators are inserted. Such a point of view of the Vassiliev invariants is expected to provide a neat perspective on the result of Bar-Natan[9]. Another importance of such a point of view comes from an application to the 4-dimensional non-perturbative canonical quantum gravity of Ashtekar[4][5]. Kauffman's path integral representation in terms of the Casimir operator insertion plays an indispensable role to investigation of the 4D quantum gravity.

This paper is organized as follows. §2 contains a brief review on Kauffman's graph invariants[17][18][20] of rigid vertex isotopy. We deal with graphs with 4-valent rigid vertices. The aim of this section is to introduce Kauffman's graph invariants of Vassiliev type[19]. In §3, we discuss a generalization of Kauffman brackets of links to those of singular links whose singularities are transverse double points. The Kauffman brackets of singular links are considered provided that they can be identified with some CS vacuum expectation values of Wilson loops. In addition to the skein relation, we introduce the spinor identity, and explain how it is used to resolve transverse double points. It is an essential role of the spinor identity that it allows us to express the Kauffman brackets of singular links by the Kauffman brackets of links. In §4, the discussions of §3 are justified in a context of representations of the quasi-triangular ribbon Hopf algebra $U_q(sl(2,C))$. We attempt to characterize the Kauffman brackets of singular links in §3 by tangle operators of rigid vertex graphs . They are perceived as a graph generalization of the tangle operators of links that was considered in the theory of quantum group invariants of 3-manifolds[23][32][33]. Our main result is that the graph invariants of Vassiliev type can be expressed by two types of tangle operators of rigid vertices which are identified with physical objects. We check that our result is compatible with an approximate expression of Kauffman's graph invariants of Vassiliev type based on a perturbative analysis of the CS quantum gauge field theory. In addition to these, we define graph invariants given by graphs composed of 4- and 6-valent vertices and attempt to find the Casimir insertion representation. The last section is devoted to physical application to the 4-dimensional quantum gravity of Ashtekar. We apply the canonical quantization and consider the loop space representation of physical wave functions given by the graph invariants. They are called spin-network states. It is verified that all the constraints of the quantum gravity with vanishing cosmological constant to specify the physical states are satisfied by the spin network states given by the graph invariants of Vassiliev type. This is a subsidiary result

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Figure 1: Generalized Reidemeister Moves generating the rigid vertex isotopy

of the present paper. This paper is an enlarged version of [15] by the physical application.

2 Rigid Vertex Graphs and Graph Invariants of Vassiliev Type

This section is devoted to a brief review on Kauffman's graph invariants that can be extended by the quantum group invariants of links. Here, we only deals with oriented graphs G whose constituents are 4-valent rigid vertices and edges and loops[17][18]. The 4-valent rigid vertex is a disk having four strings emanating from it. We shall allow a case in which G is a sum of a finite number of connected components, i.e., $G = \amalg_i G_i$. We can regard a graph G embedded into a 3-space M^3 as a generalization of a link. Two embeddings of a graph with 4-valent rigid vertices are said to be equivalent, if one is identified with the other by a rigid vertex isotopy of M^3. In the followings, we introduce the definition of the rigid vertex isotopy and graph invariants of such a kind of isotopy.

Definition 2.1 *An embedding $\phi\colon G \to M^3$ is called the* ***rigid vertex embedding*** *if for each vertex $v \in G$, a proper 2-Disk in a ball neighborhood B of $\phi(v)$ is specified such that the image of a neighborhood of v in G is contained in D. An isotopy $h_t\colon M^3 \to M^3$ between two rigid vertex embeddings ϕ_0 and ϕ_1 is called the* ***rigid vertex isotopy*** *if it carries through the ball-disk pair for each vertex of G.*

For simplicity, we proceed assuming $M^3 = S^3$ in the followings.

The rigid vertex isotopy is generated by generalized Reidemeister moves (see Fig.1). The former three moves $R1$, $R2$ and $R3$ are well-known in the theory of links. The latter two moves $R4$ and $R5$ [1]are additional ones which appear in the theory of graphs with the rigid vertices. Using a fact that link invariants are defined to be invariant under the Reidemeister moves $R1$, $R2$ and $R3$, we can introduce a huge number of classes of graph invariants of the rigid vertex isotopy in terms of resolutions of the rigid vertex. Kauffman defined such a kind of graph invariants in [17][18].

[1] $R5$ move must be replaced by another one for the Reidemeister moves generating the topological vertex isotopy[17]. The topological vertex is a point with strings emanating from it.

Suppose that we are given link invariants $P(L)$. Then, $P(L)$ can always be extended to graph invariants of the generalized Reidemeister moves. To be more precise, let's put a resolution of any one of rigid vertices of the following form [2]:

$$P\left(L^{(j)}\ \boxed{\text{EPS File croossarrow.ps not found}}\right) = aP\left(L^{(j-1)}\ \boxed{\text{EPS File pcrossarrow.ps not found}}\right) + bP\left(L^{(j-1)}\ \boxed{\text{EPS File ncrossarrow.ps not found}}\right) + cP\left(L^{(j-1)}\ \boxed{\text{EPS File unfoldingarrow.ps not found}}\right)\,, \tag{2.1}$$

where $L^{(j)}$ represents an embedding of a graph with j 4-valent rigid vertices. Thus $P\left(L^{(j)}\ \boxed{\text{EPS File croossarrow.ps not found}}\right)$ can be expressed by a sum of link invariants after the resolution of all the vertices. Then, we find $P(G) = \sum_{L\in S} a^{p(S)} b^{n(S)} c^{u(S)} P(L)$ where S represents a set of 2^j links that we obtain after the resolution of all the rigid vertices of G and $p(S)$, $n(S)$ and $u(S)$ stand for a number of positive crossings, that of negative crossings and that of unfoldings respectively.

Theorem 2.1 *Every* $P\left(L^{(j)}\ \boxed{\text{EPS File croossarrow.ps not found}}\right)$ *defined by (2.1) is invariant under the generalized Reidemeister moves.*

Proof. It is immediately proved by definition (see [17][18]). q.e.d.

We are ready to define graph invariants of Vassiliev type[19]. They are given by putting $a = 1$, $b = -1$, and $c = 0$, i.e., $P\left(L^{(j)}\ \boxed{\text{EPS File croossarrow.ps not found}}\right) = P\left(L^{(j-1)}\ \boxed{\text{EPS File pcrossarrow.ps not found}}\right) - P\left(L^{(j-1)}\ \boxed{\text{EPS File ncrossarrow.ps not found}}\right)$. They are objects that we will discuss in the following sections.

3 Spinor Identity and Kauffman Brackets

The Jones polynomial has a physical analogue in the CS quantum gauge field theory with $SU(2)$ gauge group. They are defined to be invariant under the Reidemeister moves $R1$, $R2$ and $R3$. Let $P(L)$ be the Jones polynomial of a link L and $Z(L)$ the Kauffman bracket of L. $Z(L)$ is given by a vacuum expectation value of linking Wilson loops in the CS quantum field theory[37]. The Jones polynomial differs from the Kauffman bracket only by a phase factor. To be precise, $P(L)$ is related to $Z(L)$ by $P(L) = \alpha^{-w(L)} Z(L)$. $\omega(L)$ is the writhe function given by a sum of crossing signs of the link diagram of L. It namely means $\omega(L) = \sum_{p\in\Gamma(L)} \epsilon(p)$ where $\Gamma(L)$ represents a set of all crossings of the link L and a signature ϵ takes 1 (-1) for a positive (negative) crossing (see Fig.2). According to Kauffman's notation $e(x) = exp(\frac{-i\pi}{r}x)$[19] α is set to be $e(\frac{3}{2})$. Kauffman's brackets are computable in terms of the skein relation:

$$e(\tfrac{1}{2})Z\left(L\ \boxed{\text{EPS File pcrossarrow.ps not found}}\right) - e(-\tfrac{1}{2})Z\left(L\ \boxed{\text{EPS File ncrossarrow.ps not found}}\right) = \left(e(1) - e(-1)\right) Z\left(L\ \boxed{\text{EPS File unfoldingarrow.ps not found}}\right)\,, \tag{3.1}$$

and

$$Z\left(L\ \boxed{\text{EPS File graph-a4.ps not found}}\right) = e(\tfrac{3}{2})Z\left(L\ \boxed{\text{EPS File graph-a3.ps not found}}\right),\ Z\left(L\ \boxed{\text{EPS File graph-a5.ps not found}}\right) = e(-\tfrac{3}{2})Z\left(L\ \boxed{\text{EPS File graph-a3.ps not found}}\right)\,. \tag{3.2}$$

[2]Assume that elementary subdiagrams D appearing in L_D have the blackboard framing. Furthermore, all the singular links look same outside the elementary subdiagrams.

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Figure 2: Signatures for a positive crossing, a negative crossing and a vertex

Eq.(3.2) gives the reason why $P(L)$ needs the phase factor $\alpha^{-w(L)}$. Owing to it, $P(L)$ can be invariant under the Reidemeister move $R1$. We remark that the Kauffman brackets are invariants of colored framed links (colored ribbons). Indeed, in the perturbation theory of the CS quantum gauge fields, we can observe that $Z(L)$ is subject to (3.2) assuming that all the Wilson loops are in the 2-dimensional representation of $SU(2)$. The Jones polynomials are also computable in terms of the skein relation obtained by rewriting (3.1) :

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$$e(2)P\left(L \quad\right) - e(-2)P\left(L \quad\right) = (e(1) - e(-1))P\left(L \quad\right) \tag{3.3}$$

We conventionally put an initial condition $P\left(L \quad\right) = 1$ for an unknot.

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In the followings, assume that we can identify the Kauffman brackets $Z(L)$ with the CS vacuum expectation values of Wilson loops. The Wilson loop is given by a trace of a holonomy along any one of link components L_i where $L = \Pi_i L_i$. We denote it by $W^{(i)}(L_i, A) \equiv Tr\left(U(s,s)\right) = Tr\left(Pexp\left(i\oint_{\gamma_i} A\right)\right)$. P represents the path-ordered product along a closed path $\gamma_i(s)$ $(0 \le s \le 1)$ which describes the embedding of the link component L_i. $Pexp\left(i\oint_{\gamma_i} A\right)$ is the holonomy along L_i given by a connection 1-form of a $SU(2)$-principle bundle over M^3.

Let's discuss a generalization of the Kauffman brackets of links to those of singular links whose singularities are transverse double points. First, we introduce a vacuum expectation value of N linking Wilson loops with only one transverse double point:

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$$Z\left(L^{(1)} \quad\right) \equiv \int DATr\left(U_j(s,t)U_j(t,s)\right) \cdot \Pi^N_{i\neq j} W^{(i)}(L_i, A)$$
$$\times\, exp\left(\frac{ik}{4\pi}\int Tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)\right) \tag{3.4}$$

for case 1 in which the j-th link component transversely intersects only at one point $(\gamma_j(s) = \gamma_j(t)$ and $0 \le s < t \le 1)$, and

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$$Z\left(L^{(1)} \quad\right) \equiv \int DATr\left(U_j(s,s)\right) \cdot Tr\left(U_k(t,t)\right) \cdot \Pi^N_{i\neq j,k} W^{(i)}(L_i, A)$$
$$\times\, exp\left(\frac{ik}{4\pi}\int Tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)\right) \tag{3.5}$$

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Figure 3: Casimir Insertion at a transverse double point

for case 2 in which the j-th and k-th link components transversely intersect only at one point. $\gamma_j(s)$ and $\gamma_k(t)$ describe L_j and L_k respectively and satisfy $\gamma_j(s) = \gamma_k(t)$. Another important object that we need to introduce for the latter argument is $Z\left(L^{(1)}\right)$ [EPS File C-insertionarrow.ps not found]. Let's define it as follows.

Definition 3.1 *Operation of inserting the quadratic Casimir operator at a transverse double point is called the* ***Casimir insertion*** *(see Fig.3). Suppose that N Wilson loops have only one transverse double point. The CS vacuum expectation value of N Wilson loops with the Casimir insertion is given by*

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$$Z\left(L^{(1)}\right) \equiv \int DA \sum_{a=1}^{dim(SU(2))=3} Tr\left(U_j(s,t)T_a U_j(t,s)T_a\right) \cdot \Pi_{i\neq j}^{N} W^{(i)}(L_i, A)$$
$$\times\, exp\left(\frac{ik}{4\pi}\int Tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)\right) \tag{3.6}$$

for case 1, and

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$$Z\left(L^{(1)}\right) \equiv \int DA \sum_{a=1}^{dim(SU(2))=3} Tr\left(U_j(s,s)T_a\right) \cdot Tr\left(T_a U_k(t,t)\right) \cdot \Pi_{i\neq j,k}^{N} W^{(i)}(L_i, A)$$
$$\times\, exp\left(\frac{ik}{4\pi}\int Tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)\right) \tag{3.7}$$

for case 2.

A generalization to more complicated cases in which there exist more than one transverse double point is trivial. In such more general cases, it is convenient to introduce the following notation. Let $Z(L^{(k,j-k)})$ $(0 \leq k \leq j)$ be generalized Kauffman brackets given by a singular link whose singularities are j transverse double points. The Casimir operators are inserted at $j-k$ transverse double points one by one. In the notation, $Z(L^{(0,0)})$ is nothing but the Kauffman bracket $Z(L)$ of a link L. $Z\left(L^{(1,0)}\right)$ and $Z\left(L^{(0,1)}\right)$ correspond to $Z\left(L^{(1)}\right)$ and $Z\left(L^{(1)}\right)$ respectively. Some of the generalized

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Figure 4: In the case 1, the composite path $\alpha\beta$ makes a transverse double point at the base point. In the case 2, closed paths α and β transversely intersect at the base point.

Kauffman brackets $Z\left(L^{(k,j-k)}\right)$ are not independent because of the Fierz identity, i.e., $\sum_a (T_a)_{ij}(T_a)_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl}$ ($N = 2$ for $SU(2)$). It induces a relation among them.

$$Z\left(L^{(k,j-k)} \text{ EPS File C-insertionarrow.ps not found} \right) = \frac{1}{2} Z\left(L^{(k,j-k-1)} \text{ EPS File unfoldingarrow.ps not found} \right) - \frac{1}{4} Z\left(L^{(k+1,j-k-1)} \text{ EPS File crossarrow.ps not found} \right) . \tag{3.8}$$

Regarding the generalized Kauffman brackets as vacuum expectation values of intersecting Wilson loops, we must take into account of the spinor identity in addition to the skein relation (3.1). It is satisfied by $Z\left(L^{(j,0)}\right)$ for $j \geq 1$. If there exist some Casimir insertions, we can always eliminate them using the Fierz identity (3.8). Let's explain the spinor identity below. Let A and B be elements of $SU(2)$, i.e., invertible 2×2 matrices. It follows that $Tr(A)Tr(B) = Tr(AB) + Tr(AB^{-1})$. This is derived from a fact : $\epsilon_{ab}\epsilon^{cd} = \delta_a^c\delta_b^d - \delta_a^d\delta_b^c$. The spinor identity induces a relation among the generalized Kauffman brackets:

$$Z(\alpha \cup \beta) = Z(\alpha\beta) + Z(\alpha\beta^{-1}) . \tag{3.9}$$

On the left hand side, $Z(\alpha \cup \beta)$ represents a vacuum expectation value of a product of two Wilson loops $Tr\,(U(\alpha))\,Tr\,(U(\beta))$ given by two closed paths α and β with the common base point. On the right hand side, $Z(\alpha\beta)$ ($Z(\alpha\beta^{-1})$) represents a vacuum expectation value of a Wilson loop $Tr\,(U(\alpha\beta))$ ($Tr\,(U(\alpha\beta^{-1}))$) along a composite and closed path $\alpha\beta$ ($\alpha\beta^{-1}$). (3.9) is called the spinor identity of the generalized Kauffman brackets.

We can use the spinor identity to resolve transverse double points. It is enough to consider two cases (see Fig.4). In one case (case 1), the closed paths α and β compose a closed path $\alpha\beta$ so that a transverse double point appears at their base point. Then the spinor identity takes the following form: [3]

$$Z\left(\text{EPS File unfoldingarrow.ps not found} \right) = Z\left(\text{EPS File 0crossarrow2.ps not found} \right) + Z\left(\text{EPS File unfoldingarrow3.ps not found} \right) , \tag{3.10}$$

[3] We employ a notation $Z(D)$ instead of $Z(L_D^{(j)})$ below. D represents an elementary sub-diagram of a singular link $L^{(j)}$. Any diagram depicted in the followings possesses a generalized blackboard framing. In this framing, any transverse double point should be replaced by a disk parallel to the paper.

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Figure 5: In the diagram G_a, α and β are closed paths. The composite path $\alpha\beta$ makes a transverse double point. In the diagram G_b, circles α and β transversely intersect only at their base point.

where

$$Z(\alpha\cup\beta) = Z\left(\text{EPS File unfoldingarrow1.ps not found}\right), \quad Z(\alpha\beta) = Z\left(\text{EPS File crossarrow2.ps not found}\right), \quad Z(\alpha\beta^{-1}) = Z\left(\text{EPS File unfoldingarrow3.ps not found}\right).$$

In the other case (case 2), two closed paths α and β transversely intersect at their base point. In this case, the spinor identity is of the form:

$$Z\left(\text{EPS File crossarrow3.ps not found}\right) = Z\left(\text{EPS File unfoldingarrow6.ps not found}\right) + Z\left(\text{EPS File unfoldingarrow2.ps not found}\right), \tag{3.11}$$

where

$$Z(\alpha\cup\beta) = Z\left(\text{EPS File crossarrow3.ps not found}\right), \quad Z(\alpha\beta) = Z\left(\text{EPS File unfoldingarrow6.ps not found}\right), \quad Z(\alpha\beta^{-1}) = Z\left(\text{EPS File unfoldingarrow2.ps not found}\right).$$

We should remark that a transverse double point appears in $\alpha\beta$ in the former case, and in $\alpha\cup\beta$ in the latter case.

Let's prepare a few results derived from the spinor identity necessary in the latter argument. First, let's discuss such a diagram as G_a (see Fig.5). Suppose that a closed path β has no intersection point except at the base point. Then it follows that

$$Z(\alpha\cup\beta) = Z\left(\text{EPS File graph-a2.ps not found}\right) = (e(1)+e(-1))Z\left(\text{EPS File graph-a3.ps not found}\right) \tag{3.12}$$

from the skein relation (3.1). Noticing $Z(\alpha\beta^{-1}) = Z\left(\text{EPS File graph-a3.ps not found}\right)$ and applying the spinor identity (3.10) to computation of $Z(\alpha\beta)$, we immediately obtain

$$Z(\alpha\beta) = Z\left(\text{EPS File graph-a1.ps not found}\right) = (e(1)+e(-1)-1)Z\left(\text{EPS File graph-a3.ps not found}\right). \tag{3.13}$$

Second, we consider such a diagram as G_b (see Fig.5) where two circles α and β transversely intersect at their base point. Choosing the normalization $Z\left(\boxed{\text{EPS File Circle.ps not found}}\right) = 1$, it is obvious that $Z(\alpha \cup \beta) = Z\left(\boxed{\text{EPS File graph-b1.ps not found}}\right)$, $Z(\alpha\beta) = Z\left(\boxed{\text{EPS File graph-b5.ps not found}}\right) = e(\frac{3}{2})$ and $Z(\alpha\beta^{-1}) = Z\left(\boxed{\text{EPS File graph-b6.ps not found}}\right) = e(-\frac{3}{2})$. We immediately find substituting them into (3.11)

$$Z\left(\boxed{\text{EPS File graph-b1.ps not found}}\right) = Z\left(\boxed{\text{EPS File graph-b5.ps not found}}\right) + Z\left(\boxed{\text{EPS File graph-b6.ps not found}}\right) = e(\tfrac{3}{2}) + e(-\tfrac{3}{2}) \ . \tag{3.14}$$

The two cases play an important role in proving the following proposition.

Proposition 3.1 *Let's assume that* $Z\left(L^{(1,0)}\boxed{\text{EPS File crossarrow.ps not found}}\right)$ *is uniquely expressible as a sum of* $Z\left(L^{(0,0)}\boxed{\text{EPS File pcrossarrow.ps not found}}\right)$ *and* $Z\left(L^{(0,0)}\boxed{\text{EPS File ncrossarrow.ps not found}}\right)$. *Then it must hold that*

$$Z\left(L^{(1,0)}\boxed{\text{EPS File crossarrow.ps not found}}\right) = \frac{1}{e(\frac{1}{2}) + e(-\frac{1}{2})}\left(Z\left(L^{(0,0)}\boxed{\text{EPS File pcrossarrow.ps not found}}\right) + Z\left(L^{(0,0)}\boxed{\text{EPS File ncrossarrow.ps not found}}\right)\right) \tag{3.15}$$

Proof By assumption, we can put $Z\left(L^{(1,0)}\boxed{\text{EPS File crossarrow.ps not found}}\right) = c_1 Z\left(L^{(0,0)}\boxed{\text{EPS File pcrossarrow.ps not found}}\right) + c_2 Z\left(L^{(0,0)}\boxed{\text{EPS File ncrossarrow.ps not found}}\right)$. For simplicity, taking into account of the Fierz identity (3.8), it is convenient to sort instead of it

$$e(\tfrac{1}{2})Z\left(L^{(0,0)}\boxed{\text{EPS File pcrossarrow.ps not found}}\right) + e(-\tfrac{1}{2})Z\left(L^{(0,0)}\boxed{\text{EPS File ncrossarrow.ps not found}}\right) = a_1 Z\left(L^{(0,1)}\boxed{\text{EPS File C-insertion.ps not found}}\right) + a_2 Z\left(L^{(1,0)}\boxed{\text{EPS File crossarrow.ps not found}}\right)$$
$$= \left(a_2 - \frac{a_1}{4}\right) Z\left(L^{(1,0)}\boxed{\text{EPS File pcrossarrow.ps not found}}\right) + \frac{a_1}{2} Z\left(L^{(0,0)}\boxed{\text{EPS File unfoldingarrow.ps not found}}\right) \ . \tag{3.16}$$

The determination of the coefficients a_1 and a_2 finishes the proof the proposition 3.1.

First, to determine the coefficients a_1 and a_2, let's consider a diagram like G_a depicted in Fig.5. Then (3.16) becomes

$$e(\tfrac{1}{2})Z\left(\boxed{\text{EPS File graph-a4.ps not found}}\right) + e(-\tfrac{1}{2})Z\left(\boxed{\text{EPS File graph-a5.ps not found}}\right)$$
$$= \left(a_2 - \frac{a_1}{4}\right) Z\left(\boxed{\text{EPS File graph-a1.ps not found}}\right) + \frac{a_1}{2} Z\left(\boxed{\text{EPS File graph-a2.ps not found}}\right) \ . \tag{3.17}$$

Remembering (3.2) , (3.12) and (3.13), one can find a relation between a_1 and a_2:

$$e(2) + e(-2) = \frac{a_1}{4}\left(e(1) + e(-1) + 1\right) + a_2\left(e(1) + e(-1) - 1\right) \ ,$$

or

$$a_2 = \frac{e(2) + e(-2)}{e(1) + e(-1) - 1} - \frac{1}{4}\frac{e(1) + e(-1) + 1}{e(1) + e(-1) - 1} a_1 \ . \tag{3.18}$$

What we have to do next is to determine the coefficient a_1. Let's consider the diagram G_b depicted in Fig.5. Application of the resolution (3.16) to the diagram G_b gives

$$
\begin{aligned}
&e(\tfrac{1}{2})Z\left(\text{EPS File graph-b2.ps not found}\right)+e(-\tfrac{1}{2})Z\left(\text{EPS File graph-b3.ps not found}\right)\\
&=\left(a_2-\frac{a_1}{4}\right)Z\left(\text{EPS File graph-b1.ps not found}\right)+\frac{a_1}{2}Z\left(\text{EPS File graph-b5.ps not found}\right)\ .
\end{aligned}
\tag{3.19}
$$

Three of the four terms appearing in (3.19) are already computed. It is also easy to compute the rest term, i.e., the first term on the left hand side of (3.19). After some algebra, we find

$$
\begin{aligned}
&Z\left(\text{EPS File graph-b2.ps not found}\right)=e(2)+e(-2)\ ,\quad Z\left(\text{EPS File graph-b3.ps not found}\right)=e(1)+e(-1)\ ,\\
&Z\left(\text{EPS File graph-b1.ps not found}\right)=e(\tfrac{3}{2})+e(-\tfrac{3}{2})\ ,\quad Z\left(\text{EPS File graph-b5.ps not found}\right)=e(\tfrac{3}{2})\ .
\end{aligned}
\tag{3.20}
$$

After substitution of (3.20) into (3.19), we finally obtain the solution

$$
a_1=2\left(e(1)+e(-1)-2\right)\ . \tag{3.21}
$$

Thus the coefficients a_1 and a_2 are determined. $Z\left(L^{(1,0)}\ \text{EPS File crossarrow.ps not found}\right)$ can be expressed in terms of $Z\left(L^{(0,0)}\ \text{EPS File pcrossarrow.ps not found}\right)$ and $Z\left(L^{(0,0)}\ \text{EPS File ncrossarrow.ps not found}\right)$ according to (3.16) and the skein relation (3.1). The proposition is proved. q.e.d.

For the Casimir insertion representation $Z\left(L^{(0,1)}\ \text{EPS File C-insertionarrow.ps not found}\right)$, it follows from the Fierz identity (3.8) that

$$
Z\left(L^{(0,1)}\ \text{EPS File C-insertionarrow.ps not found}\right)=\frac{1}{4\left(e(\frac{1}{2})-e(-\frac{1}{2})\right)}\left(Z\left(L^{(0,0)}\ \text{EPS File pcrossarrow.ps not found}\right)-Z\left(L^{(0,0)}\ \text{EPS File ncrossarrow.ps not found}\right)\right)\ . \tag{3.22}
$$

There is no reason of restriction to the $j=1$ case. It comes from simplicity of the computation. Indeed, we can consider more general cases in which there are more than one transverse double point. We can expect the following.

Conjecture 3.1 *Let $Z\left(L^{(j,0)}\right)$ $(j\geq 1)$ be the generalized Kauffman bracket given by a singular link with j transverse double points. It is given by resolution of any one of the j transverse double points.*

$$
Z\left(L^{(j,0)}\ \textit{EPS File crossarrow.ps not found}\right)=\frac{1}{e(\frac{1}{2})+e(-\frac{1}{2})}\left(Z\left(L^{(j-1,0)}\ \textit{EPS File pcrossarrow.ps not found}\right)+Z\left(L^{(j-1,0)}\ \textit{EPS File ncrossarrow.ps not found}\right)\right)\ . \tag{3.23}
$$

We checked that there is no contradiction for several cases. The next section justifies the conjecture in a context of the representations of the quasi-triangular ribbon Hopf algebra $U_q(sl(2,C))$.

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Figure 6: A trivalent graph

4 Penrose's Spin-network and Graph Invariants of Vassiliev Type

Let's begin with a brief introduction to Penrose's spin-network[27][28]. It is a trivalent graph whose edges are in representations of $SU(2)$ (see Fig.6). Penrose's spin-network has a strand representation in which each strand is colored by a tensor product representation of fundamental representations. In the strand representation, each trivalent vertex is coupling of three strands (Fig.7), e.g., coupling of i-, j-, k-units is expressed as

$$\text{EPS File tribox.ps not found} \equiv \text{EPS File triboxx.ps not found} \qquad (4.1)$$

The induces (i, j, k) are admissible only if $i + j + k \geq 2max(i, j, k)$ and $i + j + k \in 2Z$. Any box stands for an operator composed of skew-symmetrizers and symmetrizers. It may be projection operators onto irreducible representations of $SU(2)$ (the Yang tableau operators[27]). To be concrete, the skew-symmetrizer for n-units is

$$\text{EPS File nbox.ps not found} \equiv \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} sign(\sigma)\delta^{i_1}_{j_{\sigma(1)}}\delta^{i_2}_{j_{\sigma(2)}}\cdots\delta^{i_n}_{j_{\sigma(n)}}\ ,\ \text{where}\ \text{EPS File delta.ps not found} = \delta^i_j\ . \qquad (4.2)$$

The indices i_m, j_m $(1 \leq m \leq n)$ represent spinor indices. $\mathfrak{S}_n$ represents the symmetric group of order n. For instance, the skew-symmetrizers of two- and three-units are

$$\frac{1}{2!}\ \text{EPS File ... not found} = \text{EPS File ... not found} - \text{EPS File ... not found}\ ,\ \frac{1}{3!}\ \text{EPS File ... not found} = \text{EPS File ... not found} + \text{EPS File ... not found} + \text{EPS File ... not found} - \text{EPS File ... not found} - \text{EPS File ... not found} - \text{EPS File box10.ps not found}\ . \qquad (4.3)$$

The following is the symmetrizer which is given by n-units:

$$\text{EPS File nbbox.ps not found} \equiv \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\delta^{i_1}_{j_{\sigma(1)}}\delta^{i_2}_{j_{\sigma(2)}}\cdots\delta^{i_n}_{j_{\sigma(n)}}\ , \qquad (4.4)$$

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Figure 7: Strand representation of a trivalent vertex

The symmetrizers of two- and three-units,

$$\frac{1}{2!} \quad = \quad + \quad , \quad \frac{1}{3!} \quad = \quad + \quad + \quad + \quad + \quad + \quad . \tag{4.5}$$

Penrose's spin-network is, furthermore, specified by the spinor identity ($SU(2)$ case of Mandelstam's identity [14]),

$$\sqrt{-1}\epsilon^{ab}\sqrt{-1}\epsilon_{cd} - \delta^a_d\delta^b_c + \delta^a_c\delta^b_d = 0 \;\Rightarrow\; \quad - \quad + \quad = 0 \,. \tag{4.6}$$

Invariants of Penrose's spin-network are given by a functor from the category of trivalent graphs to the category of representations of $SU(2)$. (In general, we can define them for $SL(2,C)$ where we must take account of primed and unprimed spinors.) The invariants of Penrose's spin-network are given by contraction of tensor representations of the spinor representations. It corresponds to making scalars in the abstract tensor system (ATS[28]). Let's introduce a type (n,m)-tensor $\Gamma^{k_1\cdots k_m}_{l_1\cdots l_n}$. It is a morphism from $V^{\otimes n}$ (the tensor product of n spinors) to $V^{\otimes m}$, i.e., is in $V^{\otimes n*} \otimes V^{\otimes m}$. Especially, a morphism from V^0 (the trivial representation) to $V^{\otimes n}$ is here denoted by $\Psi^{l_1\cdots l_n}$. A morphism from $V^{\otimes m}$ to V^0 is denoted by $\Phi_{k_1\cdots k_m}$. The indices are raised or lowered by the Levi-Civita tensor. A graphical expression of invariants takes the following form:

$$\sum_{i,j} \Phi_{i_1 i_2\cdots i_{2n-1} i_{2n}} T^{i_1 i_2\cdots i_{2n-1} i_{2n}}_{j_1 j_2\cdots j_{2n-1} j_{2n}} \Psi^{j_1 j_2\cdots j_{2n-1} j_{2n}}$$

$$= \quad . \tag{4.7}$$

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Let's consider a closed spin-network as depicted in the Fig.6. The invariants are given by contraction of $T^{i_1 i_2 \cdots i_{2n-1} i_{2n}}_{j_1 j_2 \cdots j_{2n-1} j_{2n}}$ in terms of the Levi-Civita tensor. The graphical expression is

$$\sum_{i,j} \sqrt{-1}\epsilon_{i_1 i_2} \cdots \sqrt{-1}\epsilon_{i_{2n-1} i_{2n}} T^{i_1 i_2 \cdots i_{2n-1} i_{2n}}_{j_1 j_2 \cdots j_{2n-1} j_{2n}} \sqrt{-1}\epsilon^{j_1 j_2} \cdots \sqrt{-1}\epsilon^{j_{2n-1} j_{2n}}$$

$$= \boxed{\text{EPS File closedscalar.ps not found}} \quad . \tag{4.8}$$

In the theory of Penrose's spin-network, invariants are regarded as statistical partition functions associated with a discretized spacetime. In the continuum limit, they are expected to coincide with exponential functions of actions in the general relativity. Such a framework has been referred to as the Regge calculus[31].

The next subsection is devoted to the q-analog of Penrose's spin-network. We follow the Reshetikhin-Turaev construction of the 3-manifold invariants based on the quantum groups. The unit of strands is in the 2-dimensional representation of $U_q(sl(2,C))$ with q a root of unity. In the q-analog of Penrose's spin-network, we must take account of the spinor identity (4.6) in the context of the representation theory of $U_q(sl(2,C))$ (exactly, the quasi-triangular ribbon Hopf algebra given by $U_q(sl(2,C))$). We must remember the discussion in §3 in which the spinor identity can be translated into the language of the CS quantum gauge field theory. It is satisfied by vacuum expectation values of intersecting Wilson loops (see 3.9 in §2). The q-analog of Penrose's spin-network can be interpreted in many ways indeed. We here mean the q-analog by existence of unit of strands that satisfies the spinor identity in the CS quantum field theory. The spinor identity plays a crucial role in defining operators of rigid vertices of graphs. The spinor identity and the Fierz identity lead us to the Casimir insertion representation of the graph invariants of Vassiliev type. The q-analog of trivalent coupling of strands is a tedious exercise in the theory of the quantum group invariants. Such a discussion is out of the purpose of the present paper.

4.1 Reshetikhin-Traev Construction of Graph Invariants

According to the work of Reshetikhin and Turaev[32][33][35], the tangle operators are given by a functor of the category of tangles to the category of representations of the quasi-triangular ribbon Hopf algebra . The quasi-triangular ribbon Hopf algebra is specified by the Hopf algebra A, the universal R matrix and an invertible charmed element. The tangle

represents an object composed not only of a set of braiding open strings but also of pair creation and pair annihilation of them. The tangle operators of links such as the HOMFLY polynomials are composed of tangle operators of elementary tangles of link diagrams, i.e., a positive crossing, a negative crossing, a pair creation and a pair annihilation. The quantum group invariants of Reshetikhin and Turaev provided a powerful tool to classify 3-manifolds, i.e., quantum group invariants of 3-manifolds in terms of the Jones polynomials. Then Kirby and Mervin made their discussion more precise. The invariants of 3-manifolds are defined to be invariant under the Kirby-move. Such invariance comes from a fact that there are no less than one way to obtain any closed and oriented 3-manifold by the Dehn surgery along a link from S^3 .

In this sub-section, we discuss a generalization of tangle operators of links to those of rigid vertex graphs in the context of Reshetikhin-Traev construction of topological invariants. The tangle operators of graphs are also given by a functor from the category of graphs to the category of representations of the quasi-triangular ribbon Hopf algebra. The graph invariants of Vassiliev type introduced in §2 are regarded as a special class of the tangle operators of rigid vertex graphs. Our main interest is whether the graph invariant of Vassiliev type (given by the tangle operator) can be identified with any vacuum expectation value of the Wilson loops with a finite number of double points or not. In the followings, our discussion is restricted to the quasi-triangular ribbon Hopf algebra given by $U_q(sl(2,C))$. We mainly follow the work of Kirby and Mervin[23] in regard of notation and definition of the tangles and the tangle operators.

It is well known that the Jones polynomial given by a link L defers by a phase factor from the Kauffman bracket. Assuming that all link components of L are in the 2-dimensional representation of $U_q(sl(2,C))$, it follows that $P(L) = \alpha^{-\omega(L)}F(L)$. The Kauffman bracket $F(L)$ is given by a functor from the category of framed links to the category of representations of the quasi- triangular Hopf algebra [4] . On the other hand, remember that the Kauffman brackets are regarded as the CS vacuum expectation values of the Wilson loops in §3 where they are denoted by $Z(L)$. We must emphasize on a fact that the Kauffman bracket $F(L)$ is based on the quantum group and the Kauffman bracket $Z(L)$ is based on the CS path integral. In the present sub-section, they are distinguished.

The tangle operators of links can be extended to those of rigid vertex graphs in terms of resolution of rigid vertices following Kauffman's graph extension theorem (explained in §2). The graph invariants of Vassiliev type are given by $P\left(L\ \right) = P\left(L\ \right) - P\left(L\ \right)$. Let's introduce operators (Kauffman brackets) $F(G)$ of rigid vertex graphs such that $P(G) = \alpha^{-\omega(G)}F(G)$. [5] We wish to investigate how the operators $F(G)$ are related to vacuum expectation values of transversely intersecting Wilson loops. We proceed as follows. First, we aim at defining tangle operators of rigid vertices such that the graph

[4]In the Reshetikhin-Traev construction, F is referred to as a functor from the category of graphs to the category of representations of $U_q(sl(2,C))$. The link is a graph with no legs. The quantum group invariants of links are given by the morphism $F : L \to C$.

[5]$\omega(G) = \sum_{p\in\Gamma(G)} \epsilon(p)$. $\Gamma(G)$ represents a set of all crossings and rigid vertices of G. The signature for the rigid vertex takes zero.

invariants composed of them can be naturally identified with $Z\left(L^{(k,j-k)}\right)$ (introduced in §3), i.e., the Kauffman brackets given by the vacuum expectation values of the Wilson loops. Second, we attempt to express the operators $F(G)$ by them.

Let's begin with defining two types of tangle operators of rigid vertices. It will be soon shown that operators composed of them have significant properties that allow us to identify them with such CS vacuum expectation values of the Wilson loops with transverse double points as was discussed in §3.

Definition 4.1 *Let $\{L^{(k,j-k)}\}$ ($j \geq 1$ and $0 \leq k \leq j$) be a set of graphs with j 4-valent rigid vertices. Each $L^{(k,j-k)}$ has j-k vertices marked by ©. The rest of k vertices are with no mark. We introduce tangle operators $F\left(L^{(k,j-k)}\right)$. They are defined by two types of resolution of the rigid vertices. Resolution of a vertex with no mark is given by*

$$F\left(L^{(k,j-k)}\ \boxed{\text{EPS File crossarrow.ps not found}}\right) = \frac{1}{e(\frac{1}{2})+e(-\frac{1}{2})}\left(F\left(L^{(k-1,j-k)}\ \boxed{\text{EPS File pcrossarrow.ps not found}}\right) + F\left(L^{(k-1,j-k)}\ \boxed{\text{EPS File ncrossarrow.ps not found}}\right)\right), \tag{4.9}$$

and resolution of a vertex marked by © is given by

$$\begin{aligned} F\left(L^{(k,j-k)}\ \boxed{\text{EPS File C-insertionarrow.ps not found}}\right) &= \frac{1}{2}F\left(L^{(k,j-k-1)}\ \boxed{\text{EPS File unfoldingarrow.ps not found}}\right) - \frac{1}{4}F\left(L^{(k+1,j-k-1)}\ \boxed{\text{EPS File crossarrow.ps not found}}\right) \\ &= \frac{1}{4\left(e(\frac{1}{2})-e(-\frac{1}{2})\right)}\left(F\left(L^{(k,j-k-1)}\ \boxed{\text{EPS File pcrossarrow.ps not found}}\right) - F\left(L^{(k,j-k-1)}\ \boxed{\text{EPS File ncrossarrow.ps not found}}\right)\right). \end{aligned} \tag{4.10}$$

It is easily noticed that every tangle operator $F\left(L^{(k,j-k)}\right)$ can be completely determined by the tangle operators of links by iteration. Notice that (4.10) is nothing but the Fierz identity. The rest of this sub-section will be devoted to proving that $F\left(L^{(j,0)}\right)$ satisfies the spinor identity (see (3.10) and (3.11)).

Let's start with the following theorem for $j = 1$ case.

Theorem 4.1 *Let $F\left(L^{(1,0)}\right)$ be the tangle operator of a graph with only one rigid vertex. It is given by the resolution (4.9). Then it satisfies the spinor identity*

$$F\left(L^{(1,0)}\ \boxed{\text{EPS File crossarrow2.ps not found}}\right) = F\left(L^{(0,0)}\ \boxed{\text{EPS File unfoldingarrow.ps not found}}\right) - F\left(L^{(0,0)}\ \boxed{\text{EPS File unfoldingarrow4.ps not found}}\right), \tag{4.11}$$

for case 1 (see the left diagram of Fig.8) , and

$$F\left(L^{(1,0)}\ \boxed{\text{EPS File crossarrow3.ps not found}}\right) = F\left(L^{(0,0)}\ \boxed{\text{EPS File unfoldingarrow.ps not found}}\right) + F\left(L^{(0,0)}\ \boxed{\text{EPS File unfoldingarrow5.ps not found}}\right), \tag{4.12}$$

for case 2 (see the right diagram of Fig.8).

From the second terms on the right hand sides of (4.11) and (4.12) , the spinor identity is accompanied by the orientation reverse. Before proving the theorem 4.1, we need the following lemma.

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Figure 8: Two cases of resolution of a rigid vertex concerning the spinor identity are depicted. The graph of case 1 is obtained from the graph of case 2 by interchanging the two points on the rigid vertex (the disk) from which the two out-going strings emanate. These graphs have one-to-one correspondence to singular links with transverse double points.

Lemma 4.1 *Let all the link components of a link L be in the 2-dimensional representation V^2 and $k \in Z$ be larger than 2 or equal to 2. Then it follows that* [6]

$$F\left(\text{EPS File unfoldingarrow4.ps not found}\right) = \frac{1}{e(1)-e(-1)}\left(e(-\tfrac{1}{2})F\left(\text{EPS File pcrossarrow.ps not found}\right) - e(\tfrac{1}{2})F\left(\text{EPS File ncrossarrow.ps not found}\right)\right) , \tag{4.13}$$

for the second term on the right hand side of (4.11) and

$$F\left(\text{EPS File unfoldingarrow5.ps not found}\right) = \frac{1}{e(1)-e(-1)}\left(e(\tfrac{1}{2})F\left(\text{EPS File ncrossarrow.ps not found}\right) - e(-\tfrac{1}{2})F\left(\text{EPS File pcrossarrow.ps not found}\right)\right) , \tag{4.14}$$

for the second term on the right hand side of (4.12). (4.13) and (4.14) are satisfied by tangle operators of oriented links.

Proof. Let's introduce a colored coupon D [23] where D is an isomorphism $V^{2^*} \longrightarrow V^2$ under the assumption $k \geq 2$. V^{2^*} is dual to V^2. Let G_c represent a colored framed graph formed by inserting the colored coupon D and its inverse D^{-1} on K so that p extreme points of the link diagram of L are separated. K represents a link component of L with k-color where k represents a k-dimensional representation of $U_q(sl(2,C))$. There is a fact that

$$F(G_c) = (-1)^{(1-k)p} F(L) . \tag{4.15}$$

First, we prove (4.14). From the property of the quantum R matrix, the tangle operators of links must satisfy the skein relation:

$$F\left(\text{EPS File unfoldingarrow.ps not found}\right) = \frac{1}{e(1)-e(-1)}\left(e(\tfrac{1}{2})F\left(\text{EPS File pcrossarrow.ps not found}\right) - e(-\tfrac{1}{2})F\left(\text{EPS File ncrossarrow.ps not found}\right)\right) . \tag{4.16}$$

When rotated by 90 degrees, it looks like

$$F\left(\text{EPS File unfoldingarrow5.ps not found}\right) = \frac{1}{e(1)-e(-1)}\left(e(\tfrac{1}{2})F\left(\text{EPS File pcrossarrow2.ps not found}\right) - e(-\tfrac{1}{2})F\left(\text{EPS File ncrossarrow2.ps not found}\right)\right) . \tag{4.17}$$

[6] For simplicity, we employ the notation in which $F(L_D)$ is replaced by $F(D)$. D represents an elementary sub-diagram of L.

Moreover, one can rewrite (4.17) into the following form:

$$F\left(\boxed{\text{EPS File unfoldingarrow5.ps not found}}\right)=\frac{1}{e(1)-e(-1)}\left(e(\tfrac{1}{2})F\left(\boxed{\text{EPS File ncrossarrow.ps not found}}\right)-e(-\tfrac{1}{2})F\left(\boxed{\text{EPS File pcrossarrow.ps not found}}\right)\right)\ . \tag{4.18}$$

This follows from the following relations:

$$F\left(\boxed{\text{EPS File pcrossarrow2.ps not found}}\right)=F\left(\boxed{\text{EPS File DD-pcross.ps not found}}\right)=F\left(\boxed{\text{EPS File ncrossarrow.ps not found}}\right) \tag{4.19}$$

$$F\left(\boxed{\text{EPS File ncrossarrow2.ps not found}}\right)=F\left(\boxed{\text{EPS File DD-ncross.ps not found}}\right)=F\left(\boxed{\text{EPS File pcrossarrow.ps not found}}\right)\ . \tag{4.20}$$

Thus we proved (4.14).

Second, we prove (4.13). From (4.15), it follows that

$$\begin{aligned} F\left(\boxed{\text{EPS File unfoldingarrow4.ps not found}}\right) &= -F\left(\boxed{\text{EPS File D-unfolding.ps not found}}\right)\\ &= \frac{-1}{e(1)-e(-1)}\left(e(\tfrac{1}{2})F\left(\boxed{\text{EPS File D-ncross.ps not found}}\right)-e(-\tfrac{1}{2})F\left(\boxed{\text{EPS File D-pcross.ps not found}}\right)\right)\\ &= \frac{1}{e(1)-e(-1)}\left(e(-\tfrac{1}{2})F\left(\boxed{\text{EPS File pcrossarrow.ps not found}}\right)-e(\tfrac{1}{2})F\left(\boxed{\text{EPS File ncrossarrow.ps not found}}\right)\right)\ . \end{aligned} \tag{4.21}$$

In the second equality, we used the skein relation (4.17). In the last equality, we used the following relations:

$$F\left(\boxed{\text{EPS File D-ncross.ps not found}}\right)=F\left(\boxed{\text{EPS File ncrossarrow.ps not found}}\right),\ F\left(\boxed{\text{EPS File D-pcross.ps not found}}\right)=F\left(\boxed{\text{EPS File pcrossarrow.ps not found}}\right)\ , \tag{4.22}$$

(4.22) is immediately derived from (4.15) because there exist even number of extrema between the two coupons D and D^{-1} in both equations of (4.22). Thus we completed the proof of the lemma. q.e.d.

We are ready to prove the theorem 4.1.

Proof of Theorem 4.1 It is obvious from the skein relation (4.16) and the lemma 4.1. Substitute (4.13) and (4.14) into the right hand sides of (4.11) and (4.12) respectively and use (4.16). We arrive at (4.9) for the $j=1$ case . q.e.d.

Let's generalize the theorem 4.1 to cases of $j \geq 1$.

Theorem 4.2 *Let $F\left(L^{(j,0)}\right)$ for $j \geq 1$ be the tangle operator of a graph with j rigid vertices given by (4.9). It satisfies the spinor identity*

$$F\left(L^{(j,0)}\boxed{\text{EPS File crossarrow.ps not found}}\right)=F\left(L^{(j-1,0)}\boxed{\text{EPS File unfoldingarrow5.ps not found}}\right)-F\left(L^{(j-1,0)}\boxed{\text{EPS File unfoldingarrow4.ps not found}}\right)\ , \tag{4.23}$$

for the case 1 (see the left diagram of Fig.8), and

$$F\left(L^{(j,0)}\right) = F\left(L^{(j-1,0)}\right) + F\left(L^{(j-1,0)}\right), \tag{4.24}$$

for the case 2 (see the right diagram of Fig.8).

Proof. When $j = 1$, the theorem was already proved. So it is enough to assume $j \geq 2$. Suppose that all the rigid vertices of $L^{(j,0)}$ except the j-th one are resolved. Applying the definition (4.9) to the resolution, we find on the left hand side of (4.23) or (4.24),

$$F\left(L^{(j,0)}\right) = \frac{1}{\left(e(\frac{1}{2}) + e(-\frac{1}{2})\right)^{j-1}} \times \sum_{s=1}^{2^{j-1}} F\left(L_s^{(1,0)}\right), \tag{4.25}$$

where $\{L_s^{(1,0)}\}$ is a set of all graphs with only one rigid vertex obtained by resolving $j-1$ rigid vertices. It is necessary to distinguish graph types appearing in (4.23) from those appearing in (4.24) . Thus $F\left(L^{(j,0)}\right)$ is expressed by a sum of tangle operators of 2^{j-1} graphs with only one rigid vertex.

On the other hand, we can resolve all $j-1$ rigid vertices on the right hand sides of (4.23) and (4.24) in the same way. After all, they take the following forms:

$$F\left(L^{(j-1,0)}\right) - F\left(L^{(j-1,0)}\right)$$
$$= \frac{1}{\left(e(\frac{1}{2}) + e(-\frac{1}{2})\right)^{j-1}} \times \sum_{s=1}^{2^{j-1}} F\left(L_s^{(0,0)}\right) - F\left(L_s^{(0,0)}\right), \tag{4.26}$$

for the case 1, and

$$F\left(L^{(j-1,0)}\right) + F\left(L^{(j-1,0)}\right)$$
$$= \frac{1}{\left(e(\frac{1}{2}) + e(-\frac{1}{2})\right)^{j-1}} \times \sum_{s=1}^{2^{j-1}} F\left(L_s^{(0,0)}\right) + F\left(L_s^{(0,0)}\right), \tag{4.27}$$

for the case 2. We used the definition (4.9). What we only have to do is to show that for every s

$$F\left(L_s^{(1,0)}\right) = F\left(L_s^{(0,0)}\right) - F\left(L_s^{(0,0)}\right), \tag{4.28}$$

for the case 1, and

$$F\left(L_s^{(1,0)}\right) = F\left(L_s^{(0,0)}\right) + F\left(L_s^{(0,0)}\right), \tag{4.29}$$

for the case 2. The graph $L_s^{(1,0)}$ ($L_s^{(1,0)}$) and the links $L_s^{(0,0)}$ and $L_s^{(0,0)}$ ($L_s^{(0,0)}$ and $L_s^{(0,0)}$) in (4.28) ((4.29)) are identical outside the subdiagrams depicted for each s, if all the orientations of edges and loops are neglected. Thus the proof of (4.28) and

(4.29) is accomplished by repeating the proof of the theorem 4.1. q.e.d.

The inverse problem is also of interest to us. It is whether operators $F(L^{(j,0)})$ characterized by (4.23) and (4.24) must be subject to (4.9) or not. We can easily resolve it. As a consequence, it follows that the operators subject to the spinor identity must be subject to (4.9).

Thus we constructed the tangle operators $F(L^{(k,j-k)})$ which are uniquely identified with the CS vacuum expectation values of Wilson loops with j transverse double points. The Casimir operators are inserted at j-k transverse double points one by one. Of course, equivalence of $F(L^{(k,j-k)})$ and $Z(L^{(k,j-k)})$ should be verified order by order with respect to the CS coupling constant k in the perturbation theory. There is a technical limit to achieve it. Computation of terms in higher orders is extremely difficult. This is one of motivations of the present paper.

The following is now our main conclusion.

Corollary 4.1 *Let $P\left(L^{(1)}\ \text{EPS File croossarrow.ps not found}\right)$ be the graph invariant of Vassiliev type given by a graph with only one rigid vertex. Then it can be expressed by the two types of tangle operators of graphs:*

$$\begin{aligned}
P\left(L^{(1)}\ \text{EPS File crossarrow.ps not found}\right) &\equiv P\left(L^{(0)}\ \text{EPS File pcrossarrow.ps not found}\right) - P\left(L^{(0)}\ \text{EPS File ncrossarrow.ps not found}\right) \\
&= \alpha^{-\omega(L^{(1)})}(e(1)+e(-1)-1)(e(1)-e(-1)) \\
&\quad \times \left[2F\left(L^{(0,1)}\ \text{EPS File C-insertionarrow.ps not found}\right) - \frac{1}{2}\frac{e(1)+e(-1)+1}{e(1)+e(-1)-1}F\left(L^{(1,0)}\ \text{EPS File crossarrow.ps not found}\right)\right] .
\end{aligned} \tag{4.30}$$

Let's call the expression of (4.30) ***the Casimir insertion representation of the graph invariants of Vassiliev type*** *in the context of the Reshetikhin-Turaev construction of quantum group topological invariants.*

Proof. Solving (4.9) and (4.10) for $j = 1$,we immediately find

$$\begin{aligned}
F\left(L^{(0,0)}\ \text{EPS File pcrossarrow.ps not found}\right) &= \tfrac{1}{2}\left(e(\tfrac{1}{2})+e(-\tfrac{1}{2})\right)F\left(L^{(1,0)}\ \text{EPS File crossarrow.ps not found}\right) + 2\left(e(\tfrac{1}{2})-e(-\tfrac{1}{2})\right)F\left(L^{(0,1)}\ \text{EPS File C-insertionarrow.ps not found}\right) , \\
F\left(L^{(0,0)}\ \text{EPS File ncrossarrow.ps not found}\right) &= \tfrac{1}{2}\left(e(\tfrac{1}{2})+e(-\tfrac{1}{2})\right)F\left(L^{(1,0)}\ \text{EPS File crossarrow.ps not found}\right) - 2\left(e(\tfrac{1}{2})-e(-\tfrac{1}{2})\right)F\left(L^{(0,1)}\ \text{EPS File C-insertionarrow.ps not found}\right) .
\end{aligned} \tag{4.31}$$

According to

$$P\left(L^{(0)}\ \text{EPS File pcrossarrow.ps not found}\right) = \alpha^{-\omega\left(L^{(0,0)}\ \text{EPS File pcrossarrow.ps not found}\right)}F\left(L^{(0,0)}\ \text{EPS File pcrossarrow.ps not found}\right) , \quad P\left(L^{(0)}\ \text{EPS File ncrossarrow.ps not found}\right) = \alpha^{-\omega\left(L^{(0,0)}\ \text{EPS File ncrossarrow.ps not found}\right)}F\left(L^{(0,0)}\ \text{EPS File ncrossarrow.ps not found}\right) , \tag{4.32}$$

we arrive at (4.30) after substitution of (4.31) into (4.30). q.e.d.

The restriction to $j = 1$ is not crucial. More general cases of $j > 1$ can be considered in the same way and will be considered soon later. But then, expressions are more complicated.

We shall end this section with a few conclusions derived from the corollary 4.1. First, from the Casimir insertion representation (4.30) , it is obvious that the tangle operator of a rigid vertex is nothing but

$$F\left(\text{EPS File V-insertionarrow.ps not found}\right) \equiv (e(1)+e(-1)-1)(e(1)-e(-1))\left(2F\left(\text{EPS File C-insertionarrow.ps not found}\right) - \frac{1}{2}\frac{e(1)+e(-1)+1}{e(1)+e(-1)-1}F\left(\text{EPS File crossarrow.ps not found}\right)\right) . \tag{4.33}$$

where

$$F\left(\text{EPS File crossarrow.ps not found}\right) = \frac{1}{e\left(\frac{1}{2}\right)+e\left(-\frac{1}{2}\right)}\left(F\left(\text{EPS File pcrossarrow.ps not found}\right) + F\left(\text{EPS File ncrossarrow.ps not found}\right)\right) , \tag{4.34}$$

$$F\left(\text{EPS File C-insertionarrow.ps not found}\right) = \frac{1}{4\left(e\left(\frac{1}{2}\right)-e\left(-\frac{1}{2}\right)\right)}\left(F\left(\text{EPS File pcrossarrow.ps not found}\right) - F\left(\text{EPS File ncrossarrow.ps not found}\right)\right) . \tag{4.35}$$

F (EPS File pcrossarrow.ps not found) (F (EPS File ncrossarrow.ps not found)) is the tangle operator of the positive (negative) crossing. (4.33) says that F (EPS File V-insertionarrow.ps not found) is a linear combination of tangle operators of two kinds of rigid vertices, i.e., F (EPS File C-insertionarrow.ps not found) and F (EPS File crossarrow.ps not found).

Second, the graph invariant of Vassiliev type $P\left(L^{(1)}\right.$ EPS File croossarrow.ps not found $\left.\right)$ can immediately be generalized to graph invariants $P\left(L^{(j)}\right.$ EPS File croossarrow.ps not found $\left.\right)$ for $j>1$. Then it is obvious that

$$P\left(L^{(j)}\ \text{EPS File croossarrow.ps not found}\right) = \alpha^{-\omega(L^{(j)})}\tilde{F}(\bar{L})\circ\otimes_{t=1}^{j}F^{t}\left(\text{EPS File V-insertionarrow.ps not found}\right) , \tag{4.36}$$

where $\tilde{F}(\bar{L})$ represents a tangle operator given by a complement $\bar{L}$ obtained by cutting neighborhoods of the j rigid vertices out of the graph $L^{(j)}$. F^t (EPS File V-insertionarrow.ps not found) represents a tangle operator of the t-th rigid vertex.

Third, it is interesting to compare our formula (4.30) with Kauffman's approximate one computed in [19] . He found the Casimir insertion representation of the graph invariants of Vassiliev type based on the perturbative analysis of the CS quantum gauge field theory. He obtained the following approximate formula

$$P\left(L^{(1)}\ \text{EPS File croossarrow.ps not found}\right) = \alpha^{-\omega(L^{(j)})}\left(\frac{4\pi i}{k}\right)\left(Z\left(L^{(1,0)}\ \text{EPS File crossarrow.ps not found}\right) - \frac{3}{4}Z\left(L^{(0,1)}\ \text{EPS File C-insertionarrow.ps not found}\right)\right) \tag{4.37}$$

in the leading order of the CS coupling constant k. It is easy to check that this coincides with our formula (4.30). We can think of the expression (4.30) as a non-perturbative expression of Kauffman's approximate formula. We must mention a fact that Kauffman's approximate formulae can exist associated with any Lie algebra. A generalization of our argument of the present paper restricted to $U_q(sl(2,C))$ to any quasi-triangular Hopf algebra remains to be investigated.

4.2 Locally Integrable Condition and 6-Valent Graph Invariants

In the theory of the Vassiliev invariants, the Vassiliev invariants must be subject to the local integrability condition (which is often called the four term relation) for resolution

of a transverse triple point. The transverse triple point is allowed to form when there are more than one double point. The local integrability condition has been discussed to a great extent by Birman and Lin[11][25]. It is the following relation among the Vassiliev invariants of four singular links obtained by moving one of three axes along the other two axes in the vicinity of the transverse triple point:

$$v_i\left(\text{EPS File Northb.ps not found}\right)-v_i\left(\text{EPS File Southb.ps not found}\right)+v_i\left(\text{EPS File Eastb.ps not found}\right)-v_i\left(\text{EPS File Westb.ps not found}\right)=0\ , \qquad (4.38)$$

where v_i represents any one of the Vassiliev invariants of order i. The Vassiliev invariants of order i appear as coefficients in an expansion of the Jones polynomial of a link with respect to x where $q=e^x$. In accordance with (4.38), we can write the local integrability condition satisfied by the graph invariants of Vassiliev type defined in §2. It is

$$P\left(\text{EPS File North.ps not found}\right)-P\left(\text{EPS File South.ps not found}\right)+P\left(\text{EPS File East.ps not found}\right)-P\left(\text{EPS File West.ps not found}\right)=0\ . \qquad (4.39)$$

Each term on the left hand side of (4.39) is given by the tangle operator of a graph. (4.39) implies existence of a graph invariant given by the 6-valent vertex graph. We can define it as follows.

Definition 4.2 *Suppose that we are given graph invariants of Vassiliev type given by graphs with 4-valent vertices. We define the graph invariant of Vassiliev type given by the 6-valent vertex graph by*

$$\begin{aligned} P\left(\underbrace{\text{EPS File triple.ps not found}}_{G}\right) &= P\left(\underbrace{\text{EPS File North.ps not found}}_{G_1}\right)-P\left(\underbrace{\text{EPS File South.ps not found}}_{G_2}\right) \\ &= P\left(\underbrace{\text{EPS File West.ps not found}}_{G_3}\right)-P\left(\underbrace{\text{EPS File East.ps not found}}_{G_4}\right)\ . \end{aligned} \qquad (4.40)$$

The graph invariant given by the 6-valent vertex is a gap between the graph invariant of G_1 (G_3) and the graph invariant of G_2 (G_4).

In this sub-section, we attempt to show how (4.39) is satisfied in the Casimir insertion representation in which the graph invariants of Vassiliev type are identified with the vacuum expectation values of Wilson loops with transverse intersection points. We proceed assuming such identification in the followings. Let's use the Kauffman bracket $Z(G)$ instead of the tangle operator $F(G)$. In the work [19], Kauffman noticed the following

fact based on the perturbative analysis of the CS quantum gauge field theory. It said that (4.39) can be satisfied by the commutation relations of a Lie algebra[19] as far as only the Vassiliev invariants in the top raw of the actuality table[10][11] are concerned. But we wonder if such a point of view can work in general, i.e., even for all the Vassiliev invariants. We attempt to verify that the local integrability condition is satisfied by all the Vassiliev invariants using the Casimir insertion representation.

According to the definition (4.40), the Casimir insertion representation of graph invariants of the 6-valent vertex graphs is defined as follows. The 6-valent vertex can be regarded as the transverse triple point formed by two double points. From the first equality of (4.40), it is given by

$$
\begin{aligned}
P\left(\underbrace{\text{EPS File triple.ps not found}}_{G}\right) &= \alpha^{-\omega(G_1)} Z\left(\underbrace{\text{EPS File NVV.ps not found}}_{G_1}\right) - \alpha^{-\omega(G_2)} Z\left(\underbrace{\text{EPS File SVV.ps not found}}_{G_2}\right) \\
&= \alpha^{-\omega(G_1)} Z\left(\text{EPS File NorthVV.ps not found}\right) - \alpha^{-\omega(G_2)} Z\left(\text{EPS File SouthVV.ps not found}\right) .
\end{aligned}
\tag{4.41}
$$

We formed a disk spanned by three axes applying (4.31). We can use a fact that arbitrary shift of any one of axes forming the disk within the disk doesn't make any change of value of the graph invariant. It is natural from a point of view of the CS path integral. Thus we define the Casimir insertion representation of the 6-valent graph by the following limit:

$$
\begin{aligned}
P\left(\text{EPS File triple.ps not found}\right) &= lim_{\text{Disk}\to\text{point}} \\
a\alpha^{-\omega(G)} Z\left(\text{EPS File NorthVVS.ps not found}\right) &+ b\alpha^{-\omega(G)} Z\left(\text{EPS File NorthVVC.ps not found}\right) , \\
- c\alpha^{-\omega(G)} Z\left(\text{EPS File SouthVVS.ps not found}\right) &- d\alpha^{-\omega(G)} Z\left(\text{EPS File SouthVVC.ps not found}\right) .
\end{aligned}
\tag{4.42}
$$

where

$$
\begin{aligned}
a &= \frac{1}{2}\left(e(-1)+e(-2)\right), \quad b = 2\left(e(-1)-e(-2)\right) , \\
c &= \frac{1}{2}\left(e(1)+e(2)\right), \quad d = -2\left(e(2)-e(1)\right) .
\end{aligned}
\tag{4.43}
$$

The disk formed by three axes shrinks to a point in the limit. On the other hand, from the second equality of (4.40), the Casimir insertion representation of the 6-valent graph

is given by

$$P\left(\text{EPS File triple.ps not found}\right)=\alpha^{-\omega(G_3)}Z\left(\text{EPS File WVV.ps not found}\right)-\alpha^{-\omega(G_4)}Z\left(\text{EPS File EVV.ps not found}\right) \tag{4.44}$$

$$\begin{aligned}
&=\alpha^{-\omega(G_3)}Z\left(\text{EPS File WestVV.ps not found}\right)-\alpha^{-\omega(G_4)}Z\left(\text{EPS File EastVV.ps not found}\right)\\
&=lim_{\text{Disk}\rightarrow\text{point}}\\
&a\alpha^{-\omega(G)}Z\left(\text{EPS File WestVVS.ps not found}\right)+b\alpha^{-\omega(G)}Z\left(\text{EPS File WestVVC.ps not found}\right),\\
&-c\alpha^{-\omega(G)}Z\left(\text{EPS File EastVVS.ps not found}\right)-d\alpha^{-\omega(G)}Z\left(\text{EPS File EastVVC.ps not found}\right).
\end{aligned} \tag{4.45}$$

The deformations of the vertical axis to form disks make it easy to compare (4.42) and (4.45). We can easily check that they coincide. The following is available in understanding how to compute them explicitly.

As we have seen above, the Casimir insertion representation of the 6-valent graph invariant is determined by the limit of shrinking the disk to a point. Let's calculate it more explicitly as an exercise. Recall that the Casimir insertion representation of the tangle operator of the rigid vertex is given by (4.33):

$$F\left(\text{EPS File V-insertionarrow.ps not found}\right)=C_1F\left(\text{EPS File crossarrow.ps not found}\right)+C_2F\left(\text{EPS File C-insertionarrow.ps not found}\right)\ , \tag{4.46}$$

where

$$\begin{aligned}
C_1&=a-c=-\frac{1}{2}\left(e(1)-e(-1)\right)\left(e(1)+e(-1)+1\right),\\
C_2&=b-d=2\left(e(1)-e(-1)\right)\left(e(1)+e(-1)-1\right)\ .
\end{aligned} \tag{4.47}$$

The rigid vertex is replaced by a transverse double point of the Wilson loops at the moment. We can apply it to the computation. After some algebra, we find

$$\begin{aligned}
P\left(\text{EPS File triple.ps not found}\right)&=P\left(\text{EPS File North.ps not found}\right)-P\left(\text{EPS File South.ps not found}\right)\\
&=aZ\left(\text{EPS File NVVS2.ps not found}\right)+bZ\left(\text{EPS File NVVC2.ps not found}\right)\\
&\quad-cZ\left(\text{EPS File SVVS2.ps not found}\right)-dZ\left(\text{EPS File SVVC2.ps not found}\right)
\end{aligned}$$

$$= H_1 + H_2 + H_3 + H_4 + H_5\ . \tag{4.48}$$

where

$$H_1 = (C_1)^3 \alpha^{-\omega(G)} Z \left(\text{EPS File tri.ps not found} \right), \tag{4.49}$$

$$H_2 = (C_1)^2 C_2 \tag{4.50}$$
$$\times \left\{ \alpha^{-\omega(G)} Z \left(\text{EPS File triple-a.ps not found} \right) + \alpha^{-\omega(G)} Z \left(\text{EPS File triple-b.ps not found} \right) + \alpha^{-\omega(G)} Z \left(\text{EPS File triple-d.ps not found} \right) \right\},$$

$$H_3 = C_1 (C_2)^2 \tag{4.51}$$
$$\times \left\{ D^{ab}_{cd} \alpha^{-\omega(G)} Z \left(\text{EPS File triple-n.ps not found} \right) + D^{ab}_{cd} \alpha^{-\omega(G)} Z \left(\text{EPS File triple-e.ps not found} \right) + D^{ab}_{cd} \alpha^{-\omega(G)} Z \left(\text{EPS File triple-f.ps not found} \right) \right\},$$

$$H_4 = \left(C_1 C_2 (b+d) + \frac{1}{2} (C_2)^2 (a+c) \right) i \epsilon_{abc} \alpha^{-\omega(G)} Z \left(\text{EPS File triple-abc.ps not found} \right), \tag{4.52}$$

$$H_5 = (C_2)^2 \left\{ b \alpha^{-\omega(G)} Z \left(\text{EPS File triple-g.ps not found} \right) - d \alpha^{-\omega(G)} Z \left(\text{EPS File triple-h.ps not found} \right) \right\}. \tag{4.53}$$

where we used the following symmetrizer $D^{ab}_{cd} = \frac{1}{2} \left(\delta^a_c \delta^b_d + \delta^a_d \delta^b_c \right)$. ϵ_{abc} is the structure constant of the Lie algebra, i.e., $[T_a, T_b] = i \epsilon_{abc} T_c$.

Let's end this sub-section with the following result.

Remark 4.1 *We found the Casimir insertion representation of the Vassiliev invariants of graphs with 4-valent and 6-valent vertices. The local integrability condition (or the four term relation) can be satisfied by the commutation relation of the Lie algebra.*

The point in the remark has been pointed out by Bar-Natan, Kauffman et. al. from a point of view of the perturbation theory of the CS quantum gauge field theory. Thus the present work can provide a non-perturbative perspective on their observation.

5 Application to Physics

In the previous sections, we discussed quantum group invariants of rigid vertex graphs with four- and six-valent rigid vertices and found the Casimir insertion representation of them. It is an analytic expression in the CS quantum gauge field theory for the graph invariants. Importance of the Casimir insertion representation comes from mathematical and physical points of view. From the mathematical point view, it fills out a gap between an algebraic definition of the graph invariants based on the quantum groups and an analytic definition of them based on the path integral quantization of the CS field theory.

It shed light on the local integrability conditions to which the graph invariants of Vassiliev type must to be subject. On the other hand, from the physical point of view, it enables us to discuss whether the Vassiliev invariants can be physical states of the 4D quantum gravity of Ashtekar or not. The Casimir insertion representation plays a crucial role in such a discussion.

In this section, we are interested in the latter. Ashtekar's gravity provides a scheme of non-perturbative analysis in the canonical quantization. The loop space representation of wave functions in Ashtekar's gravity is most significant ,in which wave functions are defined over a loop space, i.e., a space of maps from S^1 to a 3-space. We here aim at clarifying that wave functions given by the graph invariants of Vassiliev type can be physical states in the loop space representation of the quantum gravity. Let's begin with a brief review on the canonical quantization of Ashtekar's gravity and the loop space representation of wave functions.

5.1 A Brief Review on the Loop Space Representation in the Quantum Gravity

Let $M^4 = \Sigma^3 \times R$ be a (real analytic) 4-dimensional differential manifold with a co-dimension one foliation, and $\Sigma^3(t)$ a leaf. t is a parameter of time. It is given by $t = \tau(\Sigma^3)$ in terms of a smooth map $\tau :\ \Sigma^3 \to R$. We suppose that we are given complex-valued functionals $\psi(A : \Sigma^3(t))$ defined over an affine space $\mathcal{A}$ [7] of $su(2)$-valued connection 1-forms over Σ [8] . They are sections of a line bundle over $\mathcal{A}$ specified by a set of constraints. In the canonical quantization, the constraints of Ashtekar's quantum gravity with vanishing cosmological constant take the following forms:

$$\begin{aligned}
\hat{\mathfrak{G}}[\epsilon^i]\psi(A:\Sigma) &= \int_\Sigma d^3x \epsilon^i(x)\hat{\mathcal{G}}_i\psi(A:\Sigma) \\
&= i\int_\Sigma d^3x\epsilon^i(x)\mathcal{D}_a\frac{\delta}{\delta A^i_a(x)}\psi(A:\Sigma) = 0\ , \qquad (5.1)\\
\hat{\mathfrak{M}}[N^a]\psi(A:\Sigma) &= \int_\Sigma d^3x N_a(x)\hat{\mathcal{M}}^a\psi(A:\Sigma) \\
&= i\int_\Sigma d^3x N_b(x)\frac{\delta}{\delta A^i_a(x)}F^i_{ab}\psi(A:\Sigma) = 0\ , \qquad (5.2)\\
\hat{\mathfrak{H}}[N]\psi(A:\Sigma) &= \int_\Sigma d^3x \underset{\sim}{N}(x)\hat{\mathcal{H}}\psi(A:\Sigma) \\
&= \int_\Sigma d^3x\underset{\sim}{N}(x)\epsilon^{ijk}\frac{\delta}{\delta A^i_a(x)}\frac{\delta}{\delta A^j_b(x)}F^k_{ab}\psi(A:\Sigma) = 0\ , \qquad (5.3)
\end{aligned}$$

where $\epsilon^i(x)$, the shift functions $N_a(x)$ and the lapse function $\underset{\sim}{N}(x)$ are analytic functions over M^4. The first constraint is called the Gauss law constraint, the second the momentum

[7] In Ashtekar's gravity, the self-dual connection A^4 ($A^4 = A_0dt + A_idx^i$) and the tetrad $\tilde{E}$ defined over M^4 are dynamical variables. In the (3+1)-decomposition and in $A_0 = 0$ gauge, $A \equiv A_idx^i$ is a coordinate of the configuration space on which wave functions are defined in the canonical quantization.

[8] For brevity, we replace $\Sigma^3(t)$ by Σ.

constraint which generates diffeomorphisms of Σ, the last one the Hamiltonian constraint (which is often called the Wheeler-DeWitt equation) which generates diffeomorphisms in the time direction.

Let's introduce the loop space representation of the wave functions. Remember that the Wilson loops $\mathcal{W}(\mathcal{A}:G)$ introduced in §2 are gauge invariant objects. G represents spatial graphs identified with singular links only with transverse intersection points such as transverse double and triple points. We introduce wave functions denoted by $\psi(G:\Sigma)$. They are defined by functional integration $\psi(G:\Sigma) = \int_{\mathcal{A}} d\mu(\mathcal{A})\mathcal{W}(\mathcal{A}:G)\psi(A:\Sigma)$. Constraints in the loop space representation are induced by the constraints (5.1), (5.2) and (5.3) via partial integration. To be concrete, for differential operators $\hat{\mathcal{O}} = \hat{\mathcal{M}}$ or $\hat{\mathcal{H}}$ (their adjoint operators $\hat{\mathcal{O}}^\dagger$), the constraints in the loop space representation are given by

$$\begin{aligned}\hat{\mathcal{O}}_L\psi(G:\Sigma) &= \int_{\mathcal{A}} d\mu(\mathcal{A})W(\mathcal{A}:G)\left(\hat{\mathcal{O}}\psi(A:\Sigma)\right)\\ &= \int_{\mathcal{A}} d\mu(\mathcal{A})\left(\hat{\mathcal{O}}^\dagger\mathcal{W}(\mathcal{A}:G)\right)\psi(A:\Sigma) = 0\ .\end{aligned} \tag{5.4}$$

We used partial integration to obtain the second equality. The adjoint operators are dependent on the measure. Lack of the Gauss law constraint comes from a fact that the loops space representation is defined to be manifestly gauge invariant by definition.

It is known that there is an important class of solutions to the constraint (5.4) in the loop space representation of the quantum gravity with non-zero cosmological constant. Let's consider wave functions $\psi_{cs}(A:\Sigma)$ given by the Chern-Simons action $S_{cs}(A)$, i.e., $\psi_{cs}(A:\Sigma) \equiv \exp(-\frac{6}{\Lambda}\int_\Sigma tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)) = \exp(ikS_{cs}(A))$ where $S_{cs}(A) = \frac{1}{4\pi}\int_\Sigma tr\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)$ and $\Lambda = i\frac{24\pi}{k}$ (the cosmological constant). We call them CS states. This type of solutions was initially discussed by Kodama[21]. The wave functions in the loops space representation are just Kauffman's brackets. One can easily check that they satisfy the Hamiltonian constraint with non-zero cosmological constant : $\hat{\mathcal{H}}_L^\Lambda\psi_{cs}(G:\Sigma) = \left(\hat{\mathcal{H}}_L - \frac{\Lambda}{6}\hat{det}(\tilde{E})_L\right)\psi_{cs}(G:\Sigma) = 0$. $\hat{\mathcal{H}}_L$ is induced by the Hamiltonian operator with non-zero cosmological constant:

$$\hat{\mathcal{H}}^\Lambda \equiv \epsilon^{ijk}\frac{\delta}{\delta A_a^i}\frac{\delta}{\delta A_b^j}F_{ab}^k + \frac{\Lambda}{6}\epsilon_{abc}\epsilon^{ijk}\frac{\delta}{\delta A_a^i}\frac{\delta}{\delta A_b^j}\frac{\delta}{\delta A_c^k}\ . \tag{5.5}$$

The reason why the CS states satisfy the Hamiltonian constraint is obvious from a fact:

$$\frac{\delta}{\delta A_a^i}\psi_{cs}(A:\Sigma) = i\frac{k}{8\pi}\epsilon_{abc}F_{bc}^i\psi_{cs}(A:\Sigma)\ . \tag{5.6}$$

In showing $\hat{\mathcal{H}}_L^\Lambda\psi_{cs}(G:\Sigma) = 0$, we assumed that partial integration is possible in the CS functional integral.

5.2 Spin-Network States and the Hamiltonian Constraint

The q-analog of Penrose's spin-network was initially discussed by Kauffman[20]. He arrived at the quantum group graph invariants of Dubrovnik type (a graph extension of

the Kauffman polynomials of links), that is, the quantum group invariants of unoriented graphs. However, in §2, we saw that the spinor identity which is indispensable to the q-analog of Penrose's spin-network requires the orientation of graphs. In §3, following the Reshetikhin-Turaev construction of 3-manifold invariants, we defined the q-analog of Penrose's spin-network taking into account of the orientation in an algebraic manner. The spinor identity plays a role of resulting vertices of graphs. The discussion of the graph invariants in the context of the q-analog of Penrose's spin-network led us to find the analytic expression (i.e., the Casimir insertion representation) for the graph invariants of Vassiliev type. Thus we arrived at a graph extension of the Jones polynomials. It is likely that a graph extension of the HOMFLY polynomials will be obtained by discussing Mandelstam's identity for $SU(N)$.

We here show that the graph invariants of Vassiliev type play a crucial role in the canonical quantum gravity of Ashtekar[9]. They are regarded as the physical states of the universe. Let's call them spin-network states. To be precise, they satisfy the Hamiltonian constraint with vanishing cosmological constant. The following is a conclusion of the present paper.

Theorem 5.1 *Let $P(G:\Sigma)$ be a graph invariant of Vassiliev type. G represents a singular link only with transverse double and triple points. Then it satisfies*

$$\hat{\mathcal{H}}_L P(G:\Sigma) = 0 \ . \tag{5.7}$$

Proof. It suffices to prove $\hat{\mathcal{H}}_L P(L:\Sigma) = 0$ because $P(G:\Sigma)$ can be obtained in terms of link invariants $P(L:\Sigma)$ by definition. It is obvious from the following. Let's expand the link polynomials $P(L:\Sigma)$ with respect to the inverse of the CS coupling constant $\frac{1}{k}$ or the cosmological constant Λ. Let $f_n(L)$ be a coefficient in the n-th order. It is given by a product of a group factor and a geometric factor (or an analytic factor)[1]. The Hamiltonian operator acts on the latter factor. As far as the link is concerned, the action of the Hamiltonian operator $\hat{\mathcal{H}}_L f_n(L)$ must vanish for all n because it is able to have supports only at intersection points or at kinks[12][13]. The Hamiltonian operator generates shifts of line segments along tangent vectors at the intersection points or at the kinks accompanied with recomposition of loops and partial orientation reverse. The Hamiltonian operator doesn't generate any shifts when acting on a functional of loops with no intersection points and no kinks. Thus it follows that $\hat{\mathcal{H}}_L P(G:\Sigma) = 0$. q.e.d.

Another proof comes from a fact that invariants associated to graphs with transverse triple points are calculated by the limit of shrinking a area surrounded by three axes to a point on a 2-dimensional disk. (We clarified it in §3.) But the action of the Hamiltonian operator on functionals over the space of singular links is non-trivial only at transverse triple points. So the action of the Hamiltonian operator inevitably vanishes.

[9]Importance of the Vassiliev invariants in Ashtekar's quantum gravity was first emphasized by J.Baez[8].

EPS File Frenet.ps not found

Figure 9: The Frenét frame and the complex Structure

5.3 Spin-network States and Half Flat Geometry

The spin-network states $\psi_{cs}(G:\Sigma) = P(G:\Sigma)$ are physical states in a sense that they satisfy all the constraints of the quantum gravity of Ashtekar. Our interest is physical implication of them. Remember that spin-network states are defined by functional integral of the Wilson loops with the Casimir insertion at their intersection points. In general, in the Chern-Simons path integral, i.e., contribution of flat connections over Σ dominates in the spin-network states. Flatness of the self-dual curvature $F(A)$ implies the half-flatness of the 4- dimensional geometry in Ashtekar's gravity[3]. In the connection formalism of Ashtekar's gravity with vanishing cosmological constant, Ricci flatness guarantees that the geometry is hyperkähler[34]. The hyperkähler structure is characterized by the Ricci flatness and three complex structures I, J and K with the quatanionic property, i.e., $I^2 = J^2 = K^2 = -1$ and $IJ = -JI = K$.

Let's consider the complex structures on the loop space representation of the quantum gravity. First, in a case in which loops have no intersection, it is well-known that a complex structure can be introduced in the normal bundle over a space of loops (see Fig.9). Given such a complex structure, the Frenét frame along the loops is fixed. Let N be a section of the normal bundle along a loop, then another vector B is defined by $B = T \times N = J_p N$. T is a section of the tangent bundle. The Frenét frame is given by the three vectors T, N and B. Our interest is a case of intersecting links with triple points. At each transverse triple intersection point, it is likely that we can define three almost complex structures. The three almost complex structures are introduced in three normal bundles given by three independent tangent vectors at the intersection point. However, they can not be given to be independent because the Frenét frames given by the three almost complex structures can coincide by rotations of frames and a redefinition of almost complex structures. Without lack of generality, we can put a situation that $\tilde{B} = I_p\tilde{T}$, $\tilde{N} = J_p\tilde{B}$ and $\tilde{T} = K_p\tilde{N}$. One can easily check that the almost complex structures I, J and K must satisfy a set of compatibility conditions which is just a set of quaternionic relation. Such a property as the hyperkähler structure on the space of singular links should be further explored.

Thus we obtain the following consequence:

Remark 5.1 *In the canonical quantization of Ashtekar's gravity with vanishing cosmological constant, the spin-network states given by the graph invariants of Vassiliev type*

are solutions to the Hamiltonian constraint. Furthermore, they are dominated by the half flat geometry in the classical limit.

Wave functions that we have discussed so far don't describe the physically realistic universe. The reason is that non-degenerate metrics are given only at the transverse triple points which are discrete. An idea of constructing wave functions to describe the realistic universe stems from the following inductive limit of the Vassiliev invariants, $\varinjlim E_{\infty}^{(-j.j)}$. Wave functions defined over an open dense set of transverse triple points must belong to such a group and are expected to describe the physically realistic universe.

We finally aim at construction of the quantum Hilbert space of quantum gravity of Ashtekar in our study. It may be considered either in the connection formalism or in the loop space representation. It will be constructed by a few steps. First, we define wave functions to be tri-holomorphic with respect to the three complex structures of the hyperkähler structure. Second, we introduce a physical inner product given by a hermitian pairing to be compatible with the constraints of the quantum gravity, i.e., the momentum constraint and the Hamiltonian constraint. The idea of geometric realization of the rational conformal field theories[6][7][16] will be also available in the quantization of Ashtekar's gravity. Namely,the Hamiltonian constraint will be closely related to the Kodaira-Spencer class associated to deformations of the hyperkähler structure. These things will be discussed elsewhere.

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